Five Year Integrated M.Sc. Examination, 2024
Semester - V
Course: PH-3-5-1

Time: Four Hours | Classical Mechanics | Full Marks: 80

Questions are of values as indicated in the margin
Answer question number **one** and any **four** from the rest

1. Answer any **four** questions:

$4 \times 5 = 20$

(a) What do you mean by virtual displacement? State and explain the principle of virtual work.

(b) Derive Lagrange's equation of motion from variational principle.

(c) For a particle having a Lagrangian $L = \frac{\dot{x}^2}{2x} - V(x)$, find the Hamiltonian H.

(d) If a Lagrangian $L(q, \dot{q})$ has no explicit dependence on time, prove that $\left(L - \dot{q}\frac{\partial L}{\partial \dot{q}}\right)$ is a constant of motion.

(e) Let u and v are two constants of motion. Show that Poisson's bracket of u and v i.e. $\{u, v\}$, is another constant of motion.

(f) Show that the fundamental Poisson Bracket remains invariant under canonical transformation.

2. (a) Derive Euler's equations of motion for torque free motion of a rigid body. Describe the motion of the angular velocity of the rigid body if any two principal moment of inertia of the rigid body are same.

(b) Prove that

$$\frac{dA}{dt} = \{A, H\}_{(q,p)} + \frac{\partial A}{\partial t},$$

where, A is a function of q's and p's.

(c) Calculate the following Poisson Bracket relation$\{L_x, z^2\}$, where L_x is the x component of angular momentum.

(3+3)+5+4=15

3. Suppose the potential energy of the interaction of two particles depends on the distance between them and the Lagrangian of such a system is therefore

$$\mathcal{L} = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - V(|\vec{r}_1 - \vec{r}_2|).$$

Let $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$ be the the relative position vector, and let the origin be at the centre of mass, i.e. $m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$.

(a) Show that the Lagrangian reduces to $\mathcal{L} = \frac{1}{2} m \dot{\vec{r}}^{\,2} - V(r)$ in the CM frame, where $m = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass and $r = |\vec{r}|$ is the reduced coordinate.

(b) Show that the magnitude of force associated with this potential only depends on the r, and its direction is everywhere same as that of the radius vector.

(c) Write the Lagrange equations of motion in polar coordinates, and prove that the angular momentum is a constant of motion.

(d) Using the fact that the angular momentum is a constant, show that the central force problem reduce to an one-dimensional problem.

(e) Calculate the Hamiltonian and show that the energy of the system is given by

$$E = \frac{1}{2} m \dot{r}^2 + V_{\text{eff}}(r),$$

where, $V_{\text{eff}}(r) = \frac{L^2}{2mr^2} + V(r)$ and L is the angular momentum which is a constant.

(f) Plot r vs $V_{\text{eff}}(r)$ for attractive inverse-square force i.e. $V(r) = -K/r$ $(K > 0)$ and identify the energy ranges correspond to elliptical orbit.

3+2+3+2+2+3=15

4. (a) Define Euler's angles with appropriate diagram. Why do you need three Euler's angles to describe the motion of a rigid body?

(b) Runge-Lenz vector is defined as

$$\mathbf{A} = \frac{1}{m}(\mathbf{p} \times \mathbf{L}) - \hat{\mathbf{r}}\,,$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ and $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. Show that Runge-Lenz vector is orthogonal to the angular momentum.

(c) A particle of mass m constrained to move on a vertical loop of radius R. The gravity acts downwards. Construct the Lagrangian of the system.

5+5+5=15

5. (a) Show that the following transformation

$$Q = q\cos\alpha - p\sin\alpha,$$
$$P = q\sin\alpha + p\cos\alpha,$$

is a canonical transformation for any value of the parameter α.

(b) Using Noether's theorem prove that the total linear momentum is a constant of motion if the system has translational symmetry.

(c) Determine the frequency of a simple harmonic oscillator (SHO) of mass m, force constant k by the method of action angle variables. Using it, obtain the expressions of old generalised coordinate (q) and momentum (p) in terms of canonically transformed generalised coordinate (θ) and momentum (I).(Hints: Hamiltonian of SHO is independent of θ and linearly proportional to I)

(d) Draw the phase space diagram of SHO in action-angle variables, i.e. (θ, I).

4+4+(3+3)+1=15

6. (a) Suppose the Lagrangian of a heavy symmetrical top is given by

$$L = \frac{1}{2}A(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{1}{2}C(\dot{\phi}\cos\theta + \dot{\psi})^2 - Mgl\cos\theta,$$

where θ, ϕ and ψ are three Euler's angles. Identify the cyclic coordinates and calculate the constants of motion.

(b) i. Write the energy expression for a heavy symmetrical top.

ii. Eliminating $\dot{\phi}$ and $\dot{\psi}$ from the energy expression, show that the total energy can be expressed as a function of single variable θ and its derivative.

iii. Energy expression can be rewritten as

$$\dot{u}^2 = (1-u^2)(\alpha - \beta u) - (b - au)^2 \equiv f(u)\,,$$

where, $u = \cos\theta$ and a, b, α, β are constants. Now, using $f(u)$ show that the motion of the heavy symmetrical top is confined between two boundary values of θ.

4+(1+5+5)=15